Problem One.

Suppose that $f = 0$ a.e. on E. Let E' be any set $E' \subset E$, $m(E') < \infty$, and let g be any bounded measurable function $g : E' \to \mathbb{R}$ with $0 \leq g \leq f$ (since f is nonnegative we can take g nonnegative as well).

Say that $S \subset E'$ is the set $\{x \in E' : g(x) = 0\}$. Since $f = 0$ a.e. on $E' \subset E$, and $0 \leq g \leq f$, it follows $g = 0$ a.e. on E'. Thus $m(E' - S) = 0$.

Then we have

$$\begin{aligned}\int_E f &= \sup_g \left\{ \int_{E'} g \right\} \\ &= \sup_g \left\{ \int_S g + \int_{E'-S} g \right\} \\ &= \sup_g \left\{ 0 + \int_{E'-S} g \right\} \\ &\leq \sup_g \left\{ m(E' - S) \cdot \left(\sup_{E'-S} g \right) \right\} \\ &= 0.\end{aligned}$$

For the other direction, suppose now that $\int_E f = 0$. Let $\lambda > 0$, and let $E_\lambda \subset E$ be the set $E_\lambda = \{x \in E : f(x) \geq \lambda\}$. By the Chebyshev inequality,

$$m(E_\lambda) \leq \frac{1}{\lambda} \int_E f.$$

Proof of the Chebyshev inequality is by considering the simple function $g_\lambda = \lambda \cdot \chi_{E_\lambda}$, or in the case where $m(E_\lambda) = \infty$, take a series of simple functions $g_{\lambda,n} = \lambda \chi_{E_{\lambda,n}}$ with the sets $E_{\lambda,n} = E_\lambda \cap B_n(0)$.

Now the set $T \subset E$ defined by $T = \{x \in E : f(x) > 0\}$ can be written as $T = \bigcap_\lambda E_\lambda$. Choose $\lambda = \frac{1}{n}$. Then we get in particular that

$$m(T) \leq \frac{1}{n} \int_E f = 0$$

for any $n \in \mathbb{N}$, so $m(T) = 0$. □

Remark. Many of you did this without Chebyshev inequality, considering instead that $\int_E f = 0$ implies that all simple functions g satisfying $0 \leq g = \sum_{i=1}^N y_i \chi_{S_i} \leq f$ must have integral zero, i.e., $g = 0$ a.e. on E. Then express f as the limit of a sequence of simple functions. I like this proof a lot!

Problem Two.

Fix $\epsilon > 0$. Applying Egorov to $(f_j)_{j \in \mathbb{N}}$ restricted to the domain $B_m(0) \cap E$, we can find $A_m \subset B_m(0) \cap E$ such that $m((B_m(0) \cap E) - A_m) < \frac{\epsilon}{2^m}$ and $f_j|_{A_m} \to f|_{A_m}$ uniformly as $j \to \infty$.

Now it follows that $f_j \to f$ uniformly when restricted to the domain $E_M = \bigcup_{m=1}^{M} A_m$, since the convergence is uniform on each A_m, and there are only finitely many of them. (Thus if $|f_j(x) - f(x)| < \epsilon$ when $j > J_m$ for $x \in A_m$, we can take the maximum of the J_m's, and it will work for all of E_M.)

This gives us the desired $E_0 \subset E_1 \subset \cdots \subset E$, with uniform convergence on each E_M. Now we just check that $\mu(E - \bigcup_{M=1}^{\infty} E_M) = 0$. Indeed, $E - \bigcup_{M=1}^{\infty} E_M$ turns out to be the intersection of all the bad bits $(B_m(0) \cap E) - A_m$, we can see:

$$\begin{aligned}
E - \bigcup_{M=1}^{\infty} E_M &= E \cap \left(\bigcup_{M=1}^{\infty} E_M \right)^c \\
&= E \cap \left(\bigcup_{m=1}^{\infty} A_m \right)^c \\
&= E \cap \bigcap_{m=1}^{\infty} (A_m)^c \\
&= E \cap \bigcap_{m=1}^{\infty} (B_m(0))^c \cup ((B_m(0) \cap E) - A_m) \\
&= \lim_{k \to \infty} \bigcap_{m=k}^{\infty} (B_m(0) \cap E) - A_m
\end{aligned}$$

which has measure zero. (In the final step, we used that $(B_m(0))^c$ and $((B_m(0) \cap E) - A_m)$ are disjoint, and eventually each $x \in E$ will be contained in some $B_m(0)$, thus not in any further $(B_m(0))^c$.) $\square$